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On the existence of spacelike constant mean curvature surfaces spanning two circular contours in Minkowski space

Rafael López*

Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain

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Abstract

Consider the Plateau problem for spacelike surfaces with constant mean curvature in three-dimensional Lorentz–Minkowski space $\mathbb{L}^3$ and spanning two circular axially symmetric contours in parallel planes. In this paper, we prove that rotational symmetric surfaces are the only solutions. We also give a result on uniqueness of spacelike surfaces of revolution with constant mean curvature as solutions of the exterior Dirichlet problem under a certain hypothesis at infinity.

MSC: 53C42; 53B30

Keywords: Spacelike surface; Constant mean curvature; Maximum principle; Flux formula

1. Introduction and statement of the results

Let $\mathbb{L}^3$ denote the three-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric $\langle, \rangle = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2$, where (x_1, x_2, x_3) are the canonical coordinates in $\mathbb{L}^3$. This article deals with spacelike immersed surfaces $x : \Sigma \to \mathbb{L}^3$ with constant mean curvature H. From the variational viewpoint, it is well known that such surfaces are critical points of the area functional for variations which preserve a suitable volume function. On the other hand, in relativity theory they appear in the problem of finding global time coordinates in a given spacetime [5,10].

A natural family to study consists of the surfaces of revolution. By a such surface, we mean a surface that is invariant under the action of a uniparametric subgroup of isometries of $\mathbb{L}^3$. The isometries group of $\mathbb{L}^3$ is the semi-direct product of the translations group and the orthogonal Lorentzian group $O(1, 2)$. With respect to the orthogonal group, there are three one-parameter subgroups of isometries of $\mathbb{L}^3$ depending on the causal character of the axis. In this paper we are interested by those surfaces whose axis is a timelike line and we will call these surfaces rotational symmetric surfaces again. After a Lorentzian transformation, we suppose throughout this work that this axis is the x_3-line and so, the surface is foliated by Euclidean circles in horizontal planes and centred at the axis.

* Corresponding author. Tel.: +34 958 240472; fax: +34 958 243 281.
E-mail address: rcamino@ugr.es.

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Rotational symmetric surfaces with constant mean curvature have an important role in the study of spacelike constant mean curvature of $\mathbb{L}^3$ since they can be used as barrier surfaces. For example, this occurs in the general scheme in the solvability of the Dirichlet problem for the mean curvature equation, by establishing the necessary a priori estimates. We refer the reader to [4] and [12] as examples in this context. They are also useful in the study of the singularities of a (weakly) spacelike surface. Singularities appear by the degeneracy of the ellipticity of the mean curvature equation that can drop the regularity of the metric. Rotational symmetric surfaces with constant mean curvature allow one to control the geometry of singularities. For example, Bartnik [3] proved that an isolated singular point in a spacelike surface in $\mathbb{L}^3$ with C^1 mean curvature corresponds to a regular point of the surface or a point where the surface is asymptotic to the upper (lower) light cone at this point. In the last case, the point is called a conical-type singularity.

A natural question is that of whether a spacelike surface in $\mathbb{L}^3$ with constant mean curvature inherits the symmetries of its boundary. For example, if the boundary is a round circle, it has been proved that the surface is a planar disc or a hyperbolic disc [2]. In this paper, we seek spacelike surfaces with constant mean curvature and spanning two concentric circles lying in parallel planes. Since the boundary of the surface is axially rotational, it is natural to expect this property to hold for the surface. In this context, the Alexandrov reflection method shows as a powerful tool [1]. A standard application of the Alexandrov reflection method proves that such a surface inherits the symmetries of the boundary provided the surface is included in the slab determined by the two planes containing the boundary. See also [2, Theorem 11]. For example, this occurs for a maximal surface and as a consequence of the maximum principle. However and as we will see in Fig. 2, there exist examples with pieces out of this slab. This tells us that the Alexandrov technique cannot be used to obtain the desired result.

We introduce the following notation. For real numbers a and $r > 0$, let

$$\Gamma(r, a) = \{(r\cos\theta, r\sin\theta, a) \in \mathbb{R}^3; 0 \le \theta \le 2\pi\}.$$

We then pose the following

> *Problem:* Given real constants $r, R > 0$ and $a, b, H \in \mathbb{R}$, under what conditions on r, R, a, b and H does there exist an annulus-type spacelike surface with mean curvature H and spanning $\Gamma(r, a) \cup \Gamma(R, b)$?

The spacelike property of the surface imposes that $r \neq R$. We will suppose that $r < R$. It has been proved in [4] that the Dirichlet problem for the constant mean curvature equation for H can be solved with merely the existence of a spacelike surface spanning the boundary values. In our case, this means that the cone

$$u(x_1, x_2) = \frac{b-a}{R-r}\sqrt{x_1^2 + x_2^2} - r\frac{b-a}{R-r} + a$$

is spacelike. Thus, the existence of such a solution is assured if and only if $|a - b| < R - r$.

In this paper we study the existence of a spacelike rotational symmetric surface spanning $\Gamma(r, a) \cup \Gamma(R, b)$ of the form

$$X(t, \theta) = (t\cos\theta, t\sin\theta, f(t)), \quad t \in [r, R],\ 0 \le \theta \le 2\pi \tag{1}$$

where $f \in C^0([r, R]) \cap C^2(r, R)$, and with boundary values

$$f(r) = a, \quad f(R) = b.$$

Our first result is on existence of rotational graphs bounded by $\Gamma(r, a) \cup \Gamma(R, b)$. To be exact, we have

Theorem 1. *Let* $0 < r < R < \infty$ *and* $a, b \in \mathbb{R}$. *Then the following conditions are equivalent:*

(i) *There is a rotational spacelike surface of the form (1) with constant mean curvature and spanning* $\Gamma(r, a) \cup \Gamma(R, b)$.

(ii) *The numbers* r, R, a *and* b *satisfy the condition*

$$\frac{|a-b|}{R-r} < 1.$$

Moreover, we have the following properties:

(1) *The surface can extend to be a graph over* $\mathbb{R}^2 \setminus \{(0,0)\}$.
(2) *At the origin, the surface has a singularity of conical type, except when the surface is a horizontal plane or a hyperbolic plane.*
(3) *If* $H \neq 0$, *the surface is asymptotic to a light cone at infinity.*

In particular, the uniqueness of the Dirichlet problem on bounded domains implies that the graphs obtained in [4] are surfaces of revolution provided that the boundary is rotational symmetric. Since spacelike compact surfaces are essentially graphs, we conclude (see Corollary 4)

> *Surfaces of revolution are the only compact spacelike surfaces in* $\mathbb{L}^3$ *with constant mean curvature spanning two concentric circles in parallel planes.*

We end this article with a result of uniqueness of the Dirichlet problem for the exterior of a disk, characterizing the rotational symmetric surfaces in the following sense:

Theorem 2. *Let* $r > 0$ *and* $a \in \mathbb{R}$. *Let* $u = u(x)$ *define a spacelike surface with constant mean curvature in the domain* $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2; |x| > r\}$ *such that* $u = a$ *on* $\partial\Omega$. *If*

$$\frac{|u(x)|}{|x|} \to 1 \quad \text{as } |x| \to \infty, \tag{2}$$

then u describes a surface of revolution.

The proof involves the study of the flux of a closed curve in a spacelike surface, together with an application of the maximum principle for surfaces with constant mean curvature.

This paper consists of four sections. Section 2 is a preparatory section where we will mention basic properties of compact spacelike surfaces with constant mean curvature. Section 3 will be devoted to proving Theorem 1 describing the geometric shapes of the solutions. Finally, Theorem 2 will be proved in Section 4. The main results of this work were announced in [8].

2. Preliminaries and first results

An immersion $\mathbf{x} : \Sigma \to \mathbb{L}^3$ of a smooth surface Σ is called spacelike if the induced metric on the surface is positive definite. Then $\mathbf{e_3} = (0,0,1) \in \mathbb{L}^3$ is a unit timelike vector field globally defined on $\mathbb{L}^3$, which determines a time orientation on $\mathbb{L}^3$. This allows us to choose a unique unit normal vector field N on Σ in the same time orientation as $\mathbf{e}_3$, and hence we may assume that Σ is oriented by N. We will refer to N as the future-directed Gauss map of Σ. In this article all spacelike surfaces will be oriented according to this orientation.

In Lorentz–Minkowski space there are not closed spacelike surfaces. Thus, any compact spacelike surface has non-empty boundary. If Γ is a closed curve in $\mathbb{L}^3$ and $\mathbf{x} : \Sigma \to \mathbb{L}^3$ is a spacelike immersion of a compact surface, we say that the boundary of Σ is Γ if the restriction $\mathbf{x} : \partial\Sigma \to \Gamma$ is a diffeomorphism. For spacelike surfaces, the projection $\pi : \mathbb{L}^3 \to \Pi = \{x_3 = 0\}$, $\pi(x_1, x_2, x_3) = (x_1, x_2, 0)$, is a local diffeomorphism between $\text{int}(\Sigma)$ and $\pi(\text{int}(\Sigma))$. Thus, π is an open map and $\pi(\text{int}(\Sigma))$ is a domain in Π. The compactness of Σ implies that $\pi : \Sigma \to \overline{\pi(\Sigma)}$ is a covering map. Thus, we have

Proposition 3. *Let* $\mathbf{x} : \Sigma \to \mathbb{L}^3$ *be a compact spacelike surface whose boundary* Γ *is a graph over the boundary of a domain* $\Omega \subset \mathbb{R}^2$. *Then* $\mathbf{x}(\Sigma)$ *is a graph over* Ω.

As usual, we identify $\mathbf{x}(\Sigma)$ with Σ. The mean curvature H is defined by

$$H = \frac{1}{2}\,\text{trace}\,\mathrm{d}N.$$

If Σ is the graph of a function $u(x_1, x_2)$ defined over a domain Ω, the spacelike condition implies $|Du| < 1$ and the mean curvature H is expressed by

$$(1 - |Du|^2)\sum_{i=1}^{2} D_{ii}u + \sum_{i,j=1}^{2} D_i u D_j u D_{ij} u = 2H(1 - |Du|^2)^{\frac{3}{2}}, \tag{3}$$

where the indices i, j denote the corresponding differentiations with respect to x_i, x_j. This equation can alternatively be written in divergence form:

$$\mathrm{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right)=2H.$$

This equation is of quasilinear elliptic type and the Hopf lemma can be applied. As a consequence and when the domain is bounded, we have uniqueness of solutions.

The spacelike condition for a surface of revolution $X(t,\theta)$ parametrized by (1) is equivalent to

$$f'(t)^2<1 \quad \text{for any } t,$$

and the mean curvature of $X(t,\theta)$ is

$$H(X(t,\theta))=\frac{tf''(t)+(1-f'(t)^2)f'(t)}{2t(1-f'(t)^2)^{\frac{3}{2}}}.$$

A first integral is obtained by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Ht^2-\frac{tf'(t)}{\sqrt{1-f'(t)^2}}\right)=0.$$

Thus, the quantity inside the parentheses is a constant c:

$$Ht^2-\frac{tf'(t)}{\sqrt{1-f'(t)^2}}=c. \tag{4}$$

Eq. (4) may be considered the Euler–Lagrange equation for critical points of the Minkowski area functional with respect to strictly spacelike interior variations. For completeness of the Lorentzian version, we include its proof. The argument that follows is standard. Let Σ be a spacelike surface of revolution in $\mathbb{L}^3$ obtained by rotating the curve $x_1=g(x_3)$ with respect to the x_3-axis. Assume that the profile curve has fixed endpoints $r=g(a)$ and $R=g(b)$, $r<R$. The surface area and the volume of Σ are respectively

$$A(\Sigma)=2\pi\int_r^R g(x_3)\sqrt{g'(x_3)^2-1}\,\mathrm{d}x_3, \qquad V(\Sigma)=\pi\int_r^R g(x_3)^2\,\mathrm{d}x_3.$$

We seek the surface which encloses a fixed volume $V(\Sigma)$ such that the surface area $A(\Sigma)$ is a critical point for any spacelike variation of Σ. Neglecting π in the formula, we have to extremize the functional

$$J=\int_r^R\left(2g(x_3)\sqrt{g'(x_3)^2-1}-\lambda g(x_3)^2\right)\,\mathrm{d}x_3:=\int_r^R F(g(x_3),g'(x_3))\,\mathrm{d}x_3,$$

where λ is the Lagrange multiplier. We extremize this integral noting that the integrand is independent of the variable x_3. The usual Euler–Lagrange argument says that there exists a constant k such that the function g satisfies $F-g'\frac{\partial F}{\partial g'}=k$. This gives

$$\frac{2g}{\sqrt{g'^2-1}}-\lambda g^2=k.$$

By considering $f=f(x_1)$ the inverse of the function g, the above equation can be written as

$$\frac{x_1f'}{\sqrt{1-f'^2}}-\frac{\lambda}{2}x_1^2=\frac{k}{2}$$

which coincides with (4) on taking $H=\lambda/2$ and $c=-k/2$.

3. Proof of Theorem 1

Denote $f(t) = f(t; H, c)$ a solution of (4), emphasizing, if necessary, its dependence on the values H and c. It follows that $f(t; -H, -c) = -f(t; H, c)$. Without loss of generality, we assume in this article that $H \geq 0$. The existence of the rotational symmetric surface in Theorem 1 is assured if we find a constant c and a solution $f(t; H, c)$ of

$$f'(t) = \frac{Ht^2 - c}{\sqrt{t^2 + (Ht^2 - c)^2}} \tag{5}$$

with boundary data

$$f(r; H, c) = a \quad f(R; H, c) = b. \tag{6}$$

Let us define

$$h(s; H, c) = \frac{Hs^2 - c}{\sqrt{s^2 + (Hs^2 - c)^2}}.$$

Eq. (4) is defined provided $t \neq 0$ and $h'(s)$ is a continuous function. Thus, and fixing $r > 0$, there exists a unique solution $f(t; H, c)$ of (5) with initial condition $f(r) = a$. As $|f'(t)| < 1$, the solution $f(t; H, c)$ can extend provided h is defined, that is, $f(t)$ exists at least in the interval $(0, \infty)$. We are going to move the parameter c in its range to obtain $f(R; H, c) = b$. In this case, if $f(R) = b$, there exists $\xi \in [r, R]$ such that $h(\xi)(R - r) = b - a$ and so,

$$\frac{|b - a|}{R - r} = |h(\xi)| < 1,$$

and this is a necessary condition for the existence of a spacelike surface of revolution spanning $\Gamma(r, a) \cup \Gamma(R, b)$. This proves (i) $\Rightarrow$ (ii) in Theorem 1.

We now show (ii) $\Rightarrow$ (i). First, we have for any $t \geq r$ and $H \geq 0$

$$\lim_{c \to +\infty} h(t; H, c) = -1, \quad \lim_{c \to -\infty} h(t; H, c) = 1.$$

Moreover, we know that $f(R) = a + h(\xi)(R - r)$, for some $\xi \in [r, R]$ with $\xi = \xi(R, c)$. As $\xi(x)$ remains bounded in the interval $[r, R]$ and from (5), we have

$$\lim_{c \to +\infty} f(R; H, c) = a - (R - r), \quad \lim_{c \to -\infty} f(R; H, c) = a + (R - r).$$

Because $|a - b| < R - r$ and using the dependence of parameters for the solutions, there exists a real number c such that $f(R; H, c) = b$. This yields the desired solution.

We show that at $t = 0$ the surface presents a conical-type singularity unless it is a plane or a hyperbolic plane. We have to prove that

$$\lim_{t \to 0} f'(t)^2 = 1.$$

When $c = 0$, it is possible to integrate (5): if $H = 0$, the function f is a constant, that is, the surface is a horizontal plane; if $H \neq 0$, then we obtain up to constants that $f(t) = \sqrt{1 + H^2 t^2}/H$: this surface describes a hyperbolic plane and it is regular at $t = 0$. If $c \neq 0$, from (5) we have

$$\lim_{t \to 0} f'(t) = -\frac{c}{|c|}.$$

Thus,

(1) If $c < 0$, $\lim_{t \to 0} f'(t) = 1$, and the surface is tangent to the upper light cone at $(0, 0, f(0))$.
(2) If $c > 0$, $\lim_{t \to 0} f'(t) = -1$, and the surface is tangent to the lower light cone at $(0, 0, f(0))$.

Finally, when $H \neq 0$ the surface is asymptotic to a light cone at infinity provided

$$\lim_{t \to \infty} \frac{f(t)}{t} = \pm 1.$$

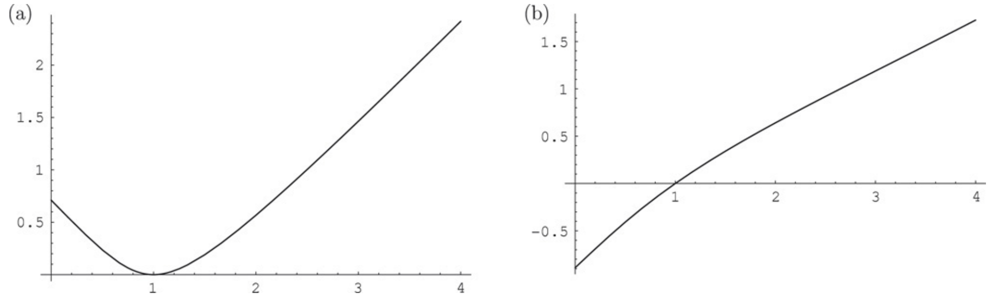

Fig. 1. Case (a): $f(t; 1, 1)$ with $f(1) = 0$. Case (b): $f(t; 0.1, -1)$ with $f(1) = 0$.

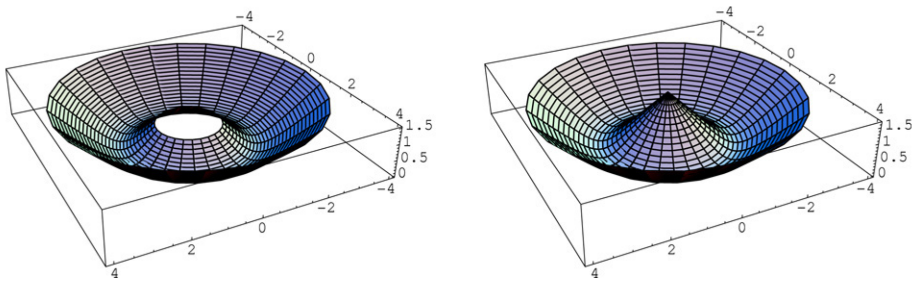

Fig. 2. In both pictures, we have a rotational symmetric surface whose profile curve is given by the function $f(t; 1, 4)$ with $f(2) = 0$. On the left, $f(t)$ is defined in the interval $[1, 4]$ with $f(1) \approx 0.65$ and $f(4) \approx 1.50$. The surface spans $\Gamma(1, 0.65) \cup \Gamma(4, 1.50)$ and it is not contained in the slab $0.65 < x_3 < 1.50$. On the right, we show the surface up to the x_3-axis.

But the l'Hôpital theorem yields

$$\lim_{t\to\infty} \frac{f(t)}{t} = \lim_{t\to\infty} f'(t) = 1.$$

If $H < 0$ this limit would be -1. We remark that when $H = 0$, $\lim_{t\to\infty} \frac{f(t)}{t} = 0$. This completes the proof of Theorem 1.

Combining Theorem 1 and Proposition 3 we obtain immediately

Corollary 4. *Any spacelike compact surface in $\mathbb{L}^3$ with constant mean curvature and spanning two concentric circles in parallel planes is a surface of revolution.*

Remark 5. A theorem due to Shiffman states that a minimal surface in Euclidean space bounded by two circles in parallel planes must be foliated by circles in parallel planes [11]. In this sense, Corollary 4 is a partial version of Shiffman's theorem in the Lorentzian setting but in our case the circles are concentric and the mean curvature is a non-zero number. On the other hand, it has been proved that a constant mean curvature spacelike surface foliated by circles is a surface of revolution (if $H \neq 0$) or it is a Lorentzian catenoid or a Riemann-type surface (if $H = 0$). See [6,7].

An easy analysis of the function $f'(t)$ such as appears in (5) gives the following result (see Fig. 1):

Corollary 6. *Let $H > 0$ and $f(t; H, c)$ be a solution of Eq. (5) defined in the interval $(0, \infty)$. Then we have:*

(1) *If $c > 0$, $f(t)$ is a convex function with a minimum at the point $t = \sqrt{c/H}$.*
(2) *If $c \leq 0$, $f(t)$ is a strictly increasing function with a unique inflection at $t = \sqrt{-c/H}$.*

As a consequence of this corollary and in the case $c > 0$, one can find examples of constant mean curvature rotational surfaces spanning two concentric contours and the surface is not included in the slab defined by the planes containing the boundary. For this, it is suffices to take an appropriate domain for $f(t; H, c)$ containing the minimum. See Fig. 2.

From the proof of Theorem 1, we can obtain a result of comparison of the profiles $f(t; H, c)$ in terms of the value c.

Corollary 7. *Assume $H \geq 0$ and fix $r > 0$ and $a \in \mathbb{R}$. Let $f(t; H, c_1)$ and $f(t; H, c_2)$ be two solutions of (5) with the same initial conditions $f(r) = a$. If $c_1 < c_2$, then $f(t; H, c_1) > f(t; H, c_2)$ for any $t > r$.*

Proof. It is immediate from the fact that the function $f'(t; H, c)$ is strictly decreasing in the variable c. □

Finally, we describe the trivial cases, that is, maximal surfaces and hyperbolic planes. Let $H = 0$. If $c = 0$, then f is a constant function and the surface is a horizontal plane. If $c \neq 0$, a simple integration leads to

$$f(t; 0, c) = c\left(\operatorname{arcsinh}\left(\frac{t}{c}\right) - \operatorname{arcsinh}\left(\frac{r}{c}\right)\right) + a.$$

Assume that $c = 0$ and $H > 0$. A direct integration gives

$$f(t; H, 0) = \frac{\sqrt{1 + H^2 t^2} - \sqrt{1 + H^2 r^2}}{H} + a.$$

The surface obtained is a domain of the hyperbolic plane $\{x \in \mathbb{L}^3; \langle x - p, x - p\rangle = -\frac{1}{H^2}\}$, where $p = (0, 0, \frac{\sqrt{1+H^2r^2}}{H})$. See Fig. 2, case (b). In particular, we have

Proposition 8. *Let a, b, r and R be such that $0 < b - a < R - r$. There exists a unique hyperbolic domain spanning $\Gamma(r, a) \cup \Gamma(R, b)$. The value of the mean curvature is*

$$H = \frac{2(b - a)}{\sqrt{\left[(R - r)^2 - (b - a)^2\right]\left[(R + r)^2 - (b - a)^2\right]}}.$$

4. Proof of Theorem 2

First, we give some remarks about the behaviour at infinity of a spacelike surface with constant mean curvature. Let $u = u(x)$, $x \in \mathbb{R}^2$, be an entire spacelike surface in $\mathbb{L}^3$ with positive constant mean curvature. Treibergs defines the projective boundary at infinity by 'blow-down' of the function u as

$$V_u(x) = \lim_{\lambda \to +\infty} \frac{u(\lambda x)}{\lambda}, \quad x \in \mathbb{R}^2.$$

He shows that V_u belongs to the class $\mathcal{Q}$ of positively convex homogeneous of degree 1 functions on $\mathbb{R}^2$ whose gradient has length 1 wherever defined. Conversely, if $W \in \mathcal{Q}$ and $H > 0$, then there is a spacelike surface u with mean curvature H such that $V_u = W$. This solution u is not unique. In fact, and when $W(x) = |x|$, he obtains many solutions. The blow-down function V_u can be defined for functions defined in $\mathbb{R}^2 \setminus B$, where B is a bounded domain, even in the case where the domain is a punctured plane. This is the case for our surfaces $f = f(t; H, c)$. A simple computation gives that $V_f = 0$ if $H = 0$, and $V_f(x) = |x|$ if $H > 0$. As a consequence, the hypothesis (2) is equivalent to $V_u(x) = |x|$, that is, the surface is asymptotic to a light cone at infinity. In this sense, Theorem 2 is a result on uniqueness provided we have two solutions with the same projective boundary at infinity and the same data in the Dirichlet condition.

In the proof of Theorem 2 we need two ingredients: the flux of a curve through a surface and the tangency principle. The flux of a surface through a closed curve is extensively used in the theory of the constant mean curvature surfaces in Euclidean space. The following notation appeared for the first time in [9]. Consider $\mathbf{x} : \Sigma \to \mathbb{L}^3$ a spacelike immersion with constant mean curvature H. Define the vectorial 1-form

$$\omega_p(v) = H\mathbf{x} \wedge d\mathbf{x}_p(v) - N(p) \wedge d\mathbf{x}_p(v), \quad p \in \Sigma$$

where v is a vector tangent to Σ at p and $\wedge$ is the cross-product in $\mathbb{L}^3$. The constancy of the mean curvature implies that ω is a closed form. It follows from the Stokes theorem that given a 1-cycle γ on Σ bounding an open $Q \subset \Sigma$, the expression

$$\mathrm{Flux}(\gamma, \Sigma) = H \int_\gamma \mathbf{x} \wedge \tau \, \mathrm{d}s - \int_\gamma \nu \, \mathrm{d}s \tag{7}$$

depends only on the homology class of γ on Σ. Here τ is a unit tangent vector of $\partial\Sigma$, and ν is the unit conormal vector of Q along γ such that $N \wedge \tau = \nu$. Note that the first integrand in (7) depends only on the curve γ. The number $\mathrm{Flux}(\gamma, \Sigma)$ is called the *flux of Σ through γ*. For example, the flux of a null-homologous cycle is zero. Formula (7) can be viewed as a measure of the forces of the surface tension of Σ that act along γ or the pressure forces that act on γ.

Given $\mathbf{e}_3 = (0, 0, 1)$, let us compute the flux on the surfaces of revolution obtained in Theorem 1 in the direction of $\mathbf{e}_3$. In what follows, we denote the number $\langle \mathrm{Flux}(\gamma, \Sigma), \mathbf{e}_3\rangle$ by $\mathrm{Flux}(\gamma, \Sigma)$ again. Fix $r > 0$. Denote as $f = f(t; H, c)$ the unique solution of (4) with initial condition $f(r) = a$. Then

$$\int_{\Gamma(r,a)} \langle \nu, \mathbf{e}_3\rangle \mathrm{d}s = \int_{\Gamma(r,a)} \frac{f'(r)}{\sqrt{1 - f'(r)^2}} \mathrm{d}s = 2\pi(Hr^2 - c)$$

and

$$H \int_{\Gamma} \langle x \wedge \tau, \mathbf{e}_3\rangle \mathrm{d}s = 2\pi H r^2.$$

Thus

$$\mathrm{Flux}(\Gamma(r, a), \Sigma) = 2\pi c.$$

As a conclusion, we have *if we fix real numbers H, r, a and λ, that there is a rotational symmetric spacelike graph Σ on $|x| > r$ spanning $\Gamma(r, a)$ with constant mean curvature H and such that* $\mathrm{Flux}(\Gamma(r, a), \Sigma) = \lambda$. Moreover, this flux is not zero, unless that Σ is a horizontal plane or a hyperbolic plane.

Finally, we state the well-known tangency principle for spacelike surfaces with constant mean curvature. Let u and v be two functions that are local expressions of two spacelike surfaces Σ_u and Σ_v of $\mathbb{L}^3$. If Σ_u and Σ_v have a common point $p = (p_1, p_2, p_3)$ where they are tangent, we will say that Σ_u lies above Σ_v near p when $u \geq v$ on a certain neighborhood of the point (p_1, p_2). The fact that (5) is of quasilinear elliptic type and the Hopf maximum principle give the following result:

Proposition 9 (*Tangency Principle*). *Let Σ_1 and Σ_2 be two spacelike surfaces (possibly with boundary) in $\mathbb{L}^3$ with the same constant mean curvature with respect to the future-directed orientation. Suppose they intersect tangentially at a point p. If Σ_1 is above Σ_2, then $\Sigma_1 = \Sigma_2$ locally around p in a neighborhood of p if one of the following hypotheses holds:*

(1) *p is an interior point of Σ_1 and Σ_2.*
(2) *p is a boundary point of Σ_1 and Σ_2 and $\partial\Sigma_1$ and $\partial\Sigma_2$ are tangent at p.*

In either case, by analyticity of solutions of elliptic equations, we conclude that Σ_1 and Σ_2 coincide whenever they are simultaneously defined.

Now, we are in position to prove Theorem 2. Denote as H the mean curvature assuming, without loss of generality, that $H \geq 0$. Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of the constant mean curvature equation (3) in Ω with $u = a$ on $\partial\Omega$. Let $\lambda = \mathrm{Flux}(\Sigma_1, \Gamma(r, a))$, where Σ_1 is the graph of u. We consider $f = f(t; H, c)$, with $t \in [r, \infty)$, $f(r) = a$ and c satisfying $\lambda = 2\pi c$. Denote as Σ_2 the graph of the function f. In particular, the fluxes of $\Gamma(r, a)$ agree in the two surfaces.

We prove that $\Sigma_1 = \Sigma_2$. We move Σ_1 upwards and define $\Sigma_1(t) = \Sigma_1 + t\mathbf{e}_3$. For $t > 0$ sufficiently big, $\Sigma_1(t)$ does not intersect Σ_2. This is possible taking into account the hypothesis (2) that says that both surfaces are asymptotic to light cones. Let us descend $\Sigma_1(t)$ by taking $t \downarrow 0$ until the first time t_0 that touches Σ_2. To be exact, let $t_0 = \inf\{t > 0; \Sigma_1(t) \cap \Sigma_2 = \emptyset\} \geq 0$.

Claim 10. *The number t_0 satisfies $t_0 = 0$.*

Proof of the Claim. We will argue by absurdity. Suppose, for absurdity, that $t_0 > 0$. Then we have two possibilities. First, if $\Sigma_1(t_0) \cap \Sigma_2 \neq \emptyset$, then both surfaces have a contact interior point. Because the two surfaces have the same mean curvature for the same (future-directed) orientation, the tangency principle implies that $\Sigma_1(t_0) = \Sigma_2$, which is impossible because they do not coincide. If $\Sigma_1(t_0) \cap \Sigma_2 = \emptyset$, then $\Sigma_1(t_0)$ and Σ_2 have a contact at infinity. Thus it is possible to choose $\epsilon > 0$ with $t_0 - \epsilon > 0$ such that at the height $t_0 - \epsilon$, the surface $\Sigma_1(t_0 - \epsilon)$ and Σ_2

intersect transversally. Then $\gamma = \Sigma_1(t_0 - \epsilon) \cap \Sigma_2$ is a one-dimensional analytic compact curve that is a graph on the x_1x_2-plane Π. We claim that the curve γ has no component null-homologous in either surface $\Sigma_1(t_0 - \epsilon)$ or Σ. On the contrary, there would be a closed curve spanning two surfaces with the same (constant) mean curvature. By the maximum principle, the two graphs must coincide, and by analyticity, $\Sigma_1(t_0 - \epsilon) = \Sigma_2$, which is impossible again. This contradiction proves the claim and thus γ is a analytic Jordan curve in the same homology class of $\Gamma(r, a)$ in both surfaces.

Let us take any component γ_0 of γ. Let Σ_1' and Σ_2' be the open sets bounded by $\gamma_0 \cup \Gamma(r, a)$ in the surfaces $\Sigma_1(t_0 - \epsilon)$ and Σ_2 respectively. Then we have

$$\mathrm{Flux}(\gamma_0, \Sigma_1') = \mathrm{Flux}(\Gamma(r, a), \Sigma_1(t_0 - \epsilon)) = (\mathrm{Flux}(\Gamma(r, a), \Sigma_2)) = (\mathrm{Flux}(\gamma_0, \Sigma_2')).$$

Since the first summands in (7) for $\mathrm{Flux}(\gamma_0, \Sigma_1')$ and $(\mathrm{Flux}(\gamma_0, \Sigma_2'))$ agree, we deduce that

$$\int_{\gamma_0} \langle \nu_1, \mathbf{e}_3 \rangle \mathrm{d}s = \int_{\gamma_0} \langle \nu_2, \mathbf{e}_3 \rangle \mathrm{d}s, \tag{8}$$

where ν_1 and ν_2 mean the unit conormal vectors on the surfaces Σ_1' and Σ_2' along γ_0, respectively. However Σ_2' lies below Σ_1' around γ_0. This can be written then as

$$\langle \nu_1, \mathbf{e}_3 \rangle \leq \langle \nu_2, \mathbf{e}_3 \rangle \quad \text{along } \gamma_0.$$

Moreover, if at some boundary point we have equality, it would imply that this point is a tangent point between the two surfaces and the tangency principle assures that the two surfaces coincide. As a conclusion, we have strict inequality $\langle \nu_1, \mathbf{e}_3 \rangle < \langle \nu_2, \mathbf{e}_3 \rangle$ along γ_0, which is an absurdity with (8). This contradiction shows that $t_0 = 0$. $\square$

As consequence of the Claim, Σ_1 goes down until having the original position at $t = 0$ and Σ_1 lies over Σ_2. Along the boundary $\Gamma(r, a)$, we have again $\langle \nu_1, \mathbf{e}_3 \rangle \leq \langle \nu_2, \mathbf{e}_3 \rangle$. If we have strict inequality at all points, we arrive at a contradiction as above, because the fluxes coincide through the two surfaces. Thus, at some boundary point $p \in \Gamma(r, a)$, we obtain $\langle \nu_1(p), \mathbf{e}_3 \rangle = \langle \nu_2(p), \mathbf{e}_3 \rangle$. As Σ_1 lies over Σ_2 around p, the tangency principle tells us that $\Sigma_1 = \Sigma_2$. This completes the proof of Theorem 2.

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